

CONTINUITY OF FUNCTIONS BELONGING TO THE FRACTIONAL ORDER SOBOLEV-GRAND LEBESGUE SPACES

E.Ostrovsky^a, L.Sirota^b

^a Corresponding Author. Department of Mathematics and computer science,
Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: galo@list.ru eugostrovsky@list.ru

^b Department of Mathematics and computer science. Bar-Ilan University, 84105,
Ramat Gan, Israel.

E - mail: sirota3@bezeqint.net

ABSTRACT.

We extend in this article the classical Sobolevs inequalities for the module of continuity for the functions belonging to the integer order Sobolev's space on the *fractional order* Sobolev - Bilateral Grand Lebesgue spaces.

As a consequence, we deduce the fractional Orlicz - Sobolev imbedding theorems and investigate the rectangle module of continuity of non-Gaussian multiparameter random fields.

Key words and phrases: Sobolev, Aronszajn, Gagliardo or Slobodeckij spaces and inequalities, imbedding theorems, weight, upper and lower estimates, module of continuity, natural function, rectangle difference, distance and module of continuity, Garsia - Rodemich - Rumsey inequality, fundamental function, Bilateral Grand Lebesgue spaces, fractional order and norm, exactness, scaling method, dilation.

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1 Notations. Statement of problem.

Let D be convex non-empty bounded closed domain with Lipschitz boundary in the whole space R^d , $d = 1, 2, \dots$, and let $f : D \rightarrow R$ be measurable function. We assume further for simplicity that $D = [0, 1]^d$. We denote $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$, $\alpha = \text{const} \in (0, 1]$,

$$|f|_p = |f|_{p,D} = \left[\int_D |f(x)|^p dx \right]^{1/p}, \quad |u(\cdot, \cdot)|_p = |u(\cdot, \cdot)|_{p,D^2} =$$

$$\left[\int_D \int_D |u(x, y)|^p dx dy \right]^{1/p}, \quad p = \text{const} \geq 1,$$

$$\omega(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in D, |x - y| \leq \delta\}, \quad \delta \in [0, \text{diam}(D)], \quad (1.0)$$

$$G_\alpha[f](x, y) = \frac{f(x) - f(y)}{|x - y|^\alpha}, \quad \nu(dx, dy) = \frac{dxdy}{|x - y|}, \quad (1.1)$$

$$|u(\cdot, \cdot)|_{p, \nu} = |u(\cdot, \cdot)|_{p, \nu, D^2} = \left[\int_D \int_D |u(x, y)|^p \nu(dx, dy) \right]^{1/p}, \quad (1.2)$$

$$\|f\|W(\alpha, p) = |G_\alpha[f](\cdot, \cdot)|_{p, \nu, D^2}. \quad (1.3)$$

The norm $\|\cdot\|W(\alpha, p)$, more precisely, semi-norm is said to be *fractional* Sobolev's norm or similar *Aronszajn, Gagliardo or Slobodeckij* norm; see, e.g. [18].

If in the definition (1.3) instead the $L_p(D^2)$ stands another norm $\|\cdot\|V(D^2)$, for instance, Lorentz, Marcinkiewicz or Grand Lebesgue, (we recall its definition further), we obtain correspondingly the definition of the fractional $\|\cdot\|V(D^2)$ norm.

The inequality

$$|f(t) - f(s)| \leq 8 \cdot 4^{1/p} \cdot \left[\frac{\alpha + 1/p}{\alpha - 1/p} \right] \cdot |t - s|^{\alpha-1/p} \cdot \|f\|W(\alpha, p), \quad (1.4)$$

or equally

$$\omega(f, \delta) \leq 8 \cdot 4^{1/p} \cdot \left[\frac{\alpha + 1/p}{\alpha - 1/p} \right] \cdot \delta^{\alpha-1/p} \cdot \left[\int_D \int_D \frac{|f(x) - f(y)|^p dxdy}{|x - y|^{\alpha p + 1}} \right]^{1/p}, \quad (1.5)$$

which is true in the case $d = 1$ (the multidimensional case will be consider further), $p > 1/\alpha$, is called *fractional* Sobolev, or Aronszajn, Gagliardo, Slobodeckij inequality.

More precisely, the inequality (1.4) implies that the function f may be redefined on the set of measure zero as a continuous function for which (1.4) there holds.

Another look on the inequality (1.4): it may be construed as an imbedding theorem from the Sobolev fractional space into the space of (uniform) continuous functions on the set D .

The proof of the our version of inequality (1.4) may be obtained immediately from an article [7], which based in turn on the famous Garsia - Rodemich - Rumsey inequality, see [6].

There are many generalizations of fractional Sobolev's imbedding theorem: on the Sobolev - Orlicz's spaces [1], p. 253-364, on the so-called *integer* Sobolev - Grand Lebesgue spaces [21], on the Lorentz and Marcinkiewicz spaces etc.

Our goal is to extend the Sobolev's imbedding theorem from integer Sobolev Grand Lebesgue spaces on the fractional Sobolev Grand Lebesgue spaces.

We recall further the definition of these spaces.

The applications of fractional Sobolev and Sobolev-Grand Lebesgue spaces in the theory of Partial Differential Equations are described, e.g. in [15], [18], [25]; in the

Functional Analysis - in [1], [6], [10], [15], [17]; in the theory of Random Processes and Fields - in [6], [7], [8], [22]; see also reference therein.

We recall in the remainder part of this section briefly the definition of the so-called Grand Lebesgue spaces; more detail investigation of these spaces see in [5], [9], [11], [14], [19], [20]; see also reference therein.

Recently appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}$, $A \geq 1$, $A < B \leq \infty$, spaces consisting on all the measurable functions $f : X \rightarrow R$ with finite norms

$$\|f\|_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \in (A, B)} \|f\|_p / \psi(p). \quad (1.6)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (A, B) = \{p : \psi(p) < \infty, \}$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [3], and are used, for example, in the theory of probability [11], [19], [20]; theory of Partial Differential Equations [5], [9]; functional analysis [5], [9], [14], [20]; theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, \infty)$ and a function $p \rightarrow p \cdot \log \psi(p)$ is convex, then the space $G\psi$ coincides with some *exponential* Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

The fundamental function of these spaces $\phi(G(\psi), \delta) = \|I_A\|_{G(\psi)}, \text{mes}(A) = \delta$, $\delta > 0$, where I_A denotes as ordinary the indicator function of the measurable set A , by the formulae

$$\phi(G(\psi), \delta) = \sup_{p \in \text{supp}(\psi)} \left[\frac{\delta^{1/p}}{\psi(p)} \right]. \quad (1.7)$$

The fundamental function of arbitrary rearrangement invariant spaces plays very important role in functional analysis, theory of Fourier series and transform [3] as well as in our further narration.

Many examples of fundamental functions for some $G\psi$ spaces are calculated in [19], [20].

Remark 1.1 If we introduce the *discontinuous* function

$$\psi_{(r)}(p) = 1, \quad p = r; \quad \psi_{(r)}(p) = \infty, \quad p \neq r, \quad p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$||f||G(\psi(r)) = |f|_r.$$

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

Remark 1.2 The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \rightarrow R$ such that $\exists(A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) \quad |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p.$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T, T$ is arbitrary set, be some *family* $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \quad \xi(t, \cdot) : X \rightarrow R$ such that

$$\exists(A, B) : 1 \leq A < B \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p.$$

The function $\psi_F(p)$ may be called as a *natural function* for the family F . This method was used in the probability theory, more exactly, in the theory of random fields, see [11],[19], chapters 3,4.

Remark 1.3 Note that the so-called *exponential* Orlicz spaces are particular cases of Grand Lebesgue spaces [11], [19], p. 34-37. In detail, let the N - Young-Orlicz function has a view

$$N(u) = e^{\mu(u)}, \tag{1.8}$$

where the function $u \rightarrow \mu(u)$ is convex even twice differentiable function such that

$$\lim_{u \rightarrow \infty} \mu'(u) = \infty.$$

Introduce a new function

$$\psi_{\{N\}}(x) = \exp \left\{ \frac{[\log N(e^x)]^*}{x} \right\}, \tag{1.9}$$

where $g^*(\cdot)$ denotes the Young-Fenchel transform of the function g :

$$g^*(x) = \sup_y (xy - g(y)).$$

Conversely, the N - function may be calculated up to equivalence through corresponding function $\psi(\cdot)$ as follows:

$$N(u) = e^{\tilde{\psi}^*(\log |u|)}, |u| > 3; N(u) = Cu^2, |u| \leq 3; \tilde{\psi}(p) = p \log \psi(p). \tag{1.10}$$

The Orlicz's space $L(N)$ over our probabilistic space is equivalent up to sublinear norms equality with Grand Lebesgue space $G\psi_{\{N\}}$.

For instance, if $N(u) = N_2(u) := \exp(u^2/2) - 1$, then $\psi_{\{N_2\}}(p) \asymp \sqrt{p}$, $p \geq 1$. The centered r.v. belonging to the Orlicz's space $L(N_2)$ are called subgaussian.

More generally, if $N(u) = N_m(u) := \exp(u^m/m) - 1$, $m = \text{const} > 0$, then $\psi_{\{N_m\}}(p) \asymp p^{1/m}$, $p \geq 1$.

Remark 1.4. The theory of probabilistic *exponential* Grand Lebesgue spaces or equally exponential Orlicz spaces gives a very convenient apparatus for investigation of the r.v. with exponential decreasing tails of distributions. Namely, the non-zero r.v. η belongs to the Orlicz space $L(N)$, where $N = N(u)$ is function described in equality (1.8), if and only if

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq \exp(-\mu(Cz)), \quad z > 1, \quad C = C(N(\cdot), \|\eta\|L(N)) \in (0, \infty). \quad (1.11)$$

(Orlicz's version).

Analogously may be written a Grand Lebesgue version of this inequality. In detail, if $0 < \|\eta\|G\psi < \infty$, then

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq 2 \exp\left(-\tilde{\psi}(\log[z/\|\eta\|G\psi])\right), \quad z \geq \|\eta\|G\psi. \quad (1.12)$$

Conversely, if

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq 2 \exp\left(-\tilde{\psi}(\log[z/K])\right), \quad z \geq K,$$

then $\|\eta\|G\psi \leq C(\psi) \cdot K$, $C(\psi) \in (0, \infty)$.

2 One dimensional result.

Let as before $\alpha = \text{const} \in (0, 1]$; the case $\alpha > 1$ in this section is trivial. We introduce the Ψ function $\zeta_\alpha(p)$ as follows:

$$\zeta_\alpha(p) := \|f\|W(\alpha, p), \quad (A, B) := \text{supp} [\zeta_\alpha(\cdot)] \quad (2.0)$$

and suppose $1 \leq A < B \leq \infty$.

Denote $A(\alpha) = \max(A, 1/\alpha)$ and suppose also

$$A(\alpha) < B. \quad (2.1)$$

Obviously, the restriction (2.1) is satisfied always in the case $B = \infty$.

We define a new psi - function $\psi_\alpha(p)$ as follows.

$$\psi_\alpha(p) := \zeta_\alpha(p) \cdot 8 \cdot 4^{1/p} \cdot \frac{\alpha + 1/p}{\alpha - 1/p}, \quad p \in (A(\alpha), B) \quad (2.2)$$

and $\psi_\alpha(p) = \infty$ otherwise.

The fractional Sobolev - Grand Lebesgue norm $\|f\|S(\alpha, \psi)$ for arbitrary function $\psi \in \Psi$ of the function $f : D \rightarrow R$ may be defined in accordance with first section up to multiplicative constant as follows: $\|f\|S(\alpha, \psi) \stackrel{\text{def}}{=}$

$$\sup_{p \in (A(\alpha), B)} \left\{ \left[8 \cdot 4^{1/p} \cdot \left(\frac{\alpha + 1/p}{\alpha - 1/p} \right) \cdot \left(\int_D \int_D \frac{|f(x) - f(y)|^p dx dy}{|x - y|^{\alpha p + 1}} \right)^{1/p} \right] / \psi(p) \right\}, \quad (2.3)$$

so that the function $\psi_\alpha(p)$ is the natural function for the function $G_\alpha(x, y)$ relative the two dimensional measure ν .

Theorem 2.1. *Let $d = 1$ and let the condition 2.1 be satisfied. Then*

$$\omega(f, \delta) \leq \frac{\delta^\alpha}{\phi(G\psi_\alpha, \delta)} \cdot \|f\| G\psi_\alpha, \quad \delta \in (0, \text{diam } D). \quad (2.4)$$

Proof. We can and will suppose without loss of generality $D = [0, 1]$ and $\|f\| S(\alpha, \psi_\alpha) = 1$. Let $p \in (A(\alpha), B)$. It follows from the definition for $\|f\| S(\alpha, \psi_\alpha)$ (2.3) that

$$8 \cdot 4^{1/p} \cdot \left[\frac{\alpha + 1/p}{\alpha - 1/p} \right] \cdot \left[\int_D \int_D \frac{|f(x) - f(y)|^p dx dy}{|x - y|^{\alpha p + 1}} \right]^{1/p} \leq \psi_\alpha(p). \quad (2.5)$$

The application of estimate (1.5) yields

$$\omega(f, \delta) \leq \delta^{\alpha-1/p} \psi_\alpha(p),$$

or equally

$$\frac{\omega(f, \delta)}{\delta^\alpha} \leq \frac{1}{\delta^{1/p} / \psi_\alpha(p)}.$$

Since the value p is arbitrary in the interval $p \in (A(\alpha), B)$, we conclude

$$\begin{aligned} \frac{\omega(f, \delta)}{\delta^\alpha} &\leq \inf_{p \in (A(\alpha), B)} \left[\frac{1}{\delta^{1/p} / \psi_\alpha(p)} \right] = \\ &= \frac{1}{\sup_{p \in (A(\alpha), B)} [\delta^{1/p} / \psi_\alpha(p)]} = \frac{1}{\phi(G(\psi_\alpha), \delta)} = \frac{\|f\| S(\alpha, \psi_\alpha)}{\phi(G(\psi_\alpha), \delta)}, \end{aligned} \quad (2.6)$$

Q.E.D.

Example 2.1. Suppose

$$\psi_\alpha(p) \asymp \psi^{(a, b; A, B)}(p) := (p - A)^{-a} (B - p)^{-b}, \quad p \in (A, B), \quad A \geq 1/\alpha, \quad a, b = \text{const} \in (0, \infty).$$

The fundamental function for the spaces $G\psi^{(a, b; A, B)}$ is investigated in [20]. Take note only that as $\delta \rightarrow 0+$

$$\phi(G\psi^{(a, b; A, B)}, \delta) \asymp \delta^{1/B} |\log \delta|^{-b}, \quad 0 < \delta < 1/e.$$

Therefore in the considered case

$$\omega(f, \delta) \leq C(\alpha, a, b; A, B) \cdot \delta^{\alpha-1/B} \cdot |\log \delta|^b \cdot \|f\| (G\psi^{(a, b; A, B)}), \quad 0 < \delta < 1/e.$$

Example 2.2. Let now

$$\psi_\alpha(p) \asymp \psi_{[\beta]}(p) := p^\beta, \quad \beta = \text{const} > 0, p > 1/\alpha.$$

We find analogously the example 2.1

$$\phi(G\psi_{[\beta]}, \delta) \asymp |\log \delta|^{-\beta}, \quad \omega(f, \delta) \leq C(\alpha, \beta) \cdot \delta^\alpha |\log \delta|^\beta \cdot \|f\| G\psi_{[\beta]}, \quad 0 < \delta < 1/e.$$

Remark 2.1. Assume in addition to the condition of theorem 2.1 $\alpha = 1$. Recall that for arbitrary rearrangement invariant space X

$$\phi(X', \delta) = \frac{\delta}{\phi(X, \delta)},$$

see [3], chapter 3; here X' denotes the associate space to the space X .

The conclusion of theorem 2.1 in the considered case $\alpha = 1$ may be rewritten as follows:

$$\omega(f, \delta) \leq \phi((G\psi)') \cdot \|f\| S(1, \psi_1), \quad \delta \in (0, \text{diam } D). \quad (2.7)$$

In general case $\alpha \in (0, 1]$ we have

$$\omega(f, \delta) \leq \delta^{\alpha-1} \cdot \phi((G\psi_\alpha)') \cdot \|f\| S(\alpha, \psi_\alpha), \quad \delta \in (0, \text{diam } D). \quad (2.8)$$

Remark 2.2. Instead the function $\nu_\alpha(p)$ may be used arbitrary it majorant.

Remark 2.3. We discuss the exactness of the assertion of theorem 2.1 further; now we proceed to the consideration of multidimensional case.

Remark 2.4. If we use in the capacity of the function $\nu_\alpha(p)$ the discontinuous function $\psi_{(r)}(p)$, we return to the inequality 1.5.

Remark 2.5. If the value α in the assertion (2.4) of theorem 2.1 is *variable* in some interval $\alpha \in (\alpha_-, \alpha_+)$, then obviously

$$\omega(f, \delta) \leq \inf_{\alpha \in (\alpha_-, \alpha_+)} \left[\frac{\delta^\alpha}{\phi(G\psi_\alpha, \delta)} \cdot \|f\| G\psi_\alpha \right], \quad \delta \in (0, \text{diam } D). \quad (2.9)$$

3 Multi-dimensional result.

The multidimensional case $d = \dim D = 2, 3, \dots$ is more complicated. Suppose for simplicity $D = [0, 1]^d$. This imply that $x \in D \Leftrightarrow x = \vec{x} = (x_1, x_2, \dots, x_d)$, $0 \leq x_i \leq 1$.

We define as in [22], [7] the *rectangle difference* operator $\square[f](\vec{x}, \vec{y}) = \square[f](x, y)$, $x, y \in D$, $f : D \rightarrow R$ as follows.

$$\Delta^{(i)}[f](x, y) := f(x_1, x_2, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_d) - f(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_d),$$

with obvious modification when $i = 1$ or $i = d$;

$$\square[f](x, y) \stackrel{def}{=} \left\{ \otimes_{i=1}^d \Delta^{(i)} \right\} [f](x, y). \quad (3.1)$$

For instance, if $d = 2$, then

$$\square[f](x, y) = f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2) + f(x_1, x_2).$$

If the function $f : [0, 1]^d \rightarrow R$ is d times continuous differentiable, then

$$\square[f](\vec{x}, \vec{y}) = \int_{x_1}^{y_1} \int_{x_2}^{y_2} \cdots \int_{x_d}^{y_d} \frac{\partial^d f}{\partial x_1 \partial x_2 \cdots \partial x_d} dx_1 dx_2 \cdots dx_d.$$

The *rectangle module of continuity* $\Omega(f, \vec{\delta}) = \Omega(f, \delta)$ for the (continuous a.e.) function f and vector $\vec{\delta} = \delta = (\delta_1, \delta_2, \dots, \delta_d) \in [0, 1]^d$ may be defined as well as ordinary module of continuity $\omega(f, \delta)$ as follows:

$$\Omega(f, \vec{\delta}) \stackrel{def}{=} \sup \{ |\square[f](x, y)|, (x, y) : |x_i - y_i| \leq \delta_i, i = 1, 2, \dots, d \}.$$

Let $\vec{\alpha} = \{\alpha_k\}$, $\alpha_k \in (0, 1]$, $k = 1, 2, \dots, d$; $p > p_0 \stackrel{def}{=} \max_k (1/\alpha_k)$, $M = \text{card}\{i, \alpha_i = \min_k \alpha_k\}$, $\delta_i = |x_i - y_i|$, $\vec{\delta} = \{\delta_i\}, i = 1, 2, \dots, d$;

$$\vec{x}^{\vec{\alpha}} := \prod_{i=1}^d x_i^{\alpha_i}, \quad \vec{\delta}^{\pm 1/p} := \left[\prod_{i=1}^d \delta_i \right]^{\pm 1/p},$$

$$G_{\vec{\alpha}}[f](x, y) = \frac{\square[f](x, y)}{|(\vec{x} - \vec{y})^{\vec{\alpha}}|}, \quad \nu(dx, dy) = \frac{d\vec{x}d\vec{y}}{|x - y|},$$

$$\|f\|W(\vec{\alpha}, p) = |G_{\vec{\alpha}}[f](\cdot, \cdot)|_{p, \nu, D^2}.$$

The norm $\|\cdot\|W(\vec{\alpha}, p)$, more precisely, semi-norm is said to be *multidimensional fractional Sobolev's norm* or similar *Aronszajn, Gagliardo or Slobodeckij* norm.

Define also as well as in the second section

$$\zeta_{\vec{\alpha}}(p) := \|f\|W(\vec{\alpha}, p), \quad (A, B) := \text{supp } [\zeta_{\vec{\alpha}}(\cdot)]$$

and suppose $1 \leq A < B \leq \infty$.

Denote $A(\vec{\alpha}) = \max(A, p_0)$ and suppose also $A(\vec{\alpha}) < B$.

We define a new psi - function $\psi_{\vec{\alpha}}(p)$ as follows.

$$\psi_{\vec{\alpha}}(p) := \zeta_{\vec{\alpha}}(p) \cdot 8^d \cdot 4^{d/p} \cdot \prod_{k=1}^d \left[\frac{\alpha_k + 1/p}{\alpha_k - 1/p} \right], \quad p \in (A(\vec{\alpha}), B).$$

Theorem 3.1. *Let $d = 1$ and let all our condition be satisfied. Then then there exists a continuous modification on the set of measure zero of the function f , which we will denote again f , for which*

$$\Omega(f, \vec{\delta}) \leq \frac{\vec{\delta}^{\vec{\alpha}}}{\phi(G\psi_{\vec{\alpha}}, \prod_{k=1}^d \delta_k)} \cdot \|f\|G\psi_{\vec{\alpha}}.$$

Proof is similar to one in theorem 2.1. We can take as before $\|f\|G\psi_{\vec{\alpha}} = 1$.

The multidimensional Garsia - Rodemich - Rumsey inequality was done by Konstantin Ral'chenko [22] (2007) at $d = 2$ and Yaozhong Hu and Khoa Le [7] (2012) in general case. Namely, let $\Phi = \Phi(y)$ be Young-Orlicz continuous even strictly increasing on the right-hand semi-axis function such that

$$\Phi(0) = 0, \quad \lim_{y \rightarrow \infty} \Phi(y) = \infty.$$

Let also $p_k = p_k(u)$, $u \in [0, 1]$, $k = 1, 2, \dots, d$ be continuous strictly increasing functions. We denote

$$B := \int \int \dots \int_{[0,1]^{2d}} \Phi \left[\frac{|\square[f](x, y)|}{\prod_{k=1}^d p_k(|x_k - y_k|)} \right] dx dy$$

and assume $B < \infty$. Then

$$|\Omega(f, \vec{\delta})| \leq 8^d \int_0^{\delta_1} \int_0^{\delta_2} \dots \int_0^{\delta_d} \Phi^{-1} \left[\frac{4^d B}{\prod_{j=1}^d u_j^2} \right] dp_1(u_1) dp_2(u_2) \dots dp_d(u_d).$$

We refer further using for us the particular case of this inequality. It asserts that for some modification of the multidimensional version of the function f

$$\begin{aligned} |\square[f](x, y)| &\leq 8^d 4^{d/p} \prod_{i=1}^d \left[\frac{\alpha_i + 1/p}{\alpha_i - 1/p} \right] \cdot \vec{\delta}^{\vec{\alpha}} \cdot \vec{\delta}^{-1/p} \times \\ &\left[\int \int \dots \int_{[0,1]^d} \frac{|\square[f](x, y)|^p}{\prod_{k=1}^d |x_k - y_k|^{\alpha_k p + 1}} dx dy \right]^{1/p} < \infty; \end{aligned} \quad (3.2)$$

or equally

$$\Omega(f, \vec{\delta}) \leq \vec{\delta}^{\vec{\alpha}} \cdot \vec{\delta}^{-1/p} \cdot \psi_{\vec{\alpha}}(p), \quad (3.3)$$

whence

$$\frac{\Omega(f, \vec{\delta})}{\vec{\delta}^{\vec{\alpha}}} \leq \frac{1}{\vec{\delta}^{1/p} / \psi_{\vec{\alpha}}(p)} = \frac{1}{[\prod_{k=1}^d \delta_k^{1/p}] / \psi_{\vec{\alpha}}(p)}. \quad (3.4)$$

Since the value p is arbitrary in the interval $p \in (A(\vec{\alpha}), B)$, we conclude

$$\begin{aligned} \frac{\Omega(f, \vec{\delta})}{\vec{\delta}^{\vec{\alpha}}} &\leq \inf_{p \in (A(\vec{\alpha}), B)} \frac{1}{[\prod_{k=1}^d \delta_k^{1/p}] / \psi_{\vec{\alpha}}(p)} = \\ &\frac{1}{\sup_{p \in (A(\vec{\alpha}), B)} [\prod_{k=1}^d \delta_k^{1/p} / \psi_{\vec{\alpha}}(p)]} = \frac{1}{\phi(G(\psi_{\vec{\alpha}}), \prod_{k=1}^d \delta_k)} = \frac{\|f\|G\psi_{\vec{\alpha}}}{\phi(G(\psi_{\vec{\alpha}}), \prod_{k=1}^d \delta_k)}, \end{aligned}$$

Q.E.D.

4 Application to the theory of random fields.

Let $\xi = \xi(x) = \xi(x_1, x_2, \dots, x_d) = \xi(\vec{x})$, $x_i \in [0, 1]$ be separable random field (r.f), not necessary to be Gaussian. The correspondent probability and expectation we will denote by \mathbf{P} , \mathbf{E} , and the probabilistic Lebesgue-Riesz L_p norm of a random variable (r.v) η we will denote as follows:

$$|\eta|_p \stackrel{def}{=} [\mathbf{E}|\eta|^p]^{1/p}.$$

We find in this section some sufficient condition for continuity of $\xi(x)$ and estimates for it *rectangle* modulus of continuity $\Omega(f, \vec{\delta})$. We apply in this section the results obtained before. Recall that the first publication about fractional Sobolev's inequalities [6] was devoted in particular to the such a problem; see also articles [7], [22].

Let us introduce the following natural Ψ function: $\theta_{\vec{\alpha}}(p) =$

$$\theta_{\alpha}(p) = 8^d \cdot 4^{d/p} \cdot \prod_{k=1}^d \left[\frac{\alpha_k + 1/p}{\alpha_k - 1/p} \right] \cdot \left[\int_0^1 \int_0^1 \mathbf{E} |G_{\vec{\alpha}}[\xi](x, y)|^p \nu(dx, dy) \right]^{1/p}, \quad (4.1)$$

$$\alpha = \vec{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_d\}, \quad \alpha_k = \text{const} > 0;$$

and suppose the function $\theta_{\alpha}(p)$ has non-trivial support such that

$$A = \inf \sup \theta_{\alpha}(\cdot) \geq 1/\min_k \alpha_k, \quad B = \sup \sup \theta_{\alpha} \in (A, \infty].$$

Theorem 4.1.

$$|\Omega[\xi], \vec{\delta}|_A \leq \frac{\vec{\delta}^{\vec{\alpha}}}{\phi(G\theta_{\alpha}, \prod_{k=1}^d \delta_k)}. \quad (4.2)$$

Proof. We use the inequality (3.2):

$$\begin{aligned} |\square[\xi](x, y)| &\leq 8^d 4^{d/p} \prod_{i=1}^d \left[\frac{\alpha_i + 1/p}{\alpha_i - 1/p} \right] \cdot \vec{\delta}^{\vec{\alpha}} \cdot \vec{\delta}^{-1/p} \times \\ &\left[\int \int \dots \int_{[0,1]^d} \frac{|\square[\xi](x, y)|^p}{\prod_{k=1}^d |x_k - y_k|^{\alpha_k p + 1}} dx dy \right]^{1/p} \end{aligned} \quad (4.3)$$

or equally

$$\begin{aligned} |\square[\xi](x, y)|^p &\leq 8^{dp} 4^d \prod_{i=1}^d \left[\frac{\alpha_i + 1/p}{\alpha_i - 1/p} \right]^p \cdot \vec{\delta}^{\vec{\alpha}p} \cdot \vec{\delta}^{-1} \times \\ &\left[\int \int \dots \int_{[0,1]^d} \frac{|\square[\xi](x, y)|^p}{\prod_{k=1}^d |x_k - y_k|^{\alpha_k p + 1}} dx dy \right]. \end{aligned} \quad (4.4)$$

We get taking expectation:

$$|\square[\xi], \vec{\delta}|_p \leq \vec{\delta}^{\vec{\alpha}} \cdot \vec{\delta}^{-1/p} \cdot \theta_\alpha(p). \quad (4.5)$$

We intend to take the infimum of both-side inequalities (4.5) over p ; $p \in (A, B)$. Note that

$$\inf_{p \in (A, B)} |\square[\xi], \vec{\delta}|_p = |\square[\xi], \vec{\delta}|_A$$

(Lyapunov's inequality) and

$$\inf_{p \in (A, B)} \vec{\delta}^{\vec{\alpha}} \cdot \vec{\delta}^{-1/p} \cdot \theta_\alpha(p) = \vec{\delta}^{\vec{\alpha}} \cdot \frac{1}{\sup_{p \in (A, B)} (\prod_{k=1}^d \delta_k)^{1/p} \theta_\alpha(p)} = \frac{\vec{\delta}^{\vec{\alpha}}}{\phi(G\theta_\alpha, \prod_{k=1}^d \delta_k)}. \quad (4.6)$$

Q.E.D.

Remark 4.1. We can obtain the exponential bounds for the tail of distribution of r.v. $\Omega[\xi], \vec{\delta}$ as follows. Let us define the so-called *truncated* fundamental function

$$\phi_q(G\psi, \delta) = \sup_{p \in (q, B)} \frac{\delta^{1/p}}{\psi(p)}, \quad A < q < B.$$

Denote also

$$\lambda(q, \delta) = \frac{1}{\phi_q(G\theta_{\vec{\alpha}}, \delta)}.$$

It follows from the proposition of theorem 4.1 that

$$\frac{\Omega[\xi], (\vec{\delta})}{\vec{\delta}^{\vec{\alpha}}} \leq \lambda(q, \prod_{k=1}^d \delta_k),$$

or equally

$$\left\| \frac{\Omega[\xi], (\vec{\delta})}{\vec{\delta}^{\vec{\alpha}}} \right\| G\lambda(\cdot, \prod_{k=1}^d \delta_k) \leq 1.$$

It remains to use the conclusion of remark 1.4.

Theorem 4.2. Suppose that for some finite positive constants $K, \alpha, \{\beta_k\}$, $k = 1, 2, \dots, d$

$$\mathbf{E}|\square[\xi](x, y)|^\alpha \leq K \cdot \prod_{k=1}^d |x_k - y_k|^{1+\beta_k}. \quad (4.7)$$

Then there exists a non-negative random variable τ with finite moment of order α : $\mathbf{E}\tau^\alpha \leq 1$ such that

$$\Omega[\xi](\vec{\delta}) \leq C(\alpha, \vec{\beta}, d) \cdot \tau \cdot K^{1/\alpha} \cdot \prod_{k=1}^d \left[\delta_k^{\beta_k/\alpha} |\log \delta_k|^{1/\alpha} \right], \quad 0 < \delta_k \leq 1/e. \quad (4.8)$$

Proof. We can assume $K = 1$ and use the multidimensional Garsia-Rodemich-Rumsey inequality, in which we choose

$$\Phi(x) = |x|^\alpha, \quad p_k(x) = |x|^{\gamma_k}, \quad 2/\alpha < \gamma_k < (2 + \beta_k)/\alpha, \quad \gamma_k = (2 + \beta_k)/\alpha - \epsilon_k. \quad (4.9)$$

Let us introduce the following random variable (r.v.)

$$B = \int \int \dots \int_{[0,1]^{2d}} \left| \frac{|\square[\xi](x, y)|}{\prod_{k=1}^d |x_k - y_k|^{\gamma_k}} \right|^\alpha dx dy. \quad (4.10)$$

We have using polar coordinates:

$$\begin{aligned} \mathbf{E}B &\leq \int \int \dots \int_{[0,1]^{2d}} \frac{\prod_{k=1}^d |x_k - y_k|^{1+\beta_k}}{\prod_{k=1}^d |x_k - y_k|^{2+\beta_k-\gamma_k}} dx dy \leq \\ &C_2(\alpha, \vec{\beta}, d) \prod_{k=1}^d \int_0^{\sqrt{d}} r_k^{-1+\alpha\epsilon_k} dr_k = C_3(\alpha, \vec{\beta}, d) \prod_{k=1}^d (1/\epsilon_k). \end{aligned} \quad (4.11)$$

Therefore the r.v. B may be represented as a product

$$B = C_3(\alpha, \vec{\beta}, d) \tau^{1/\alpha} \prod_{k=1}^d (1/\epsilon_k), \quad (4.12)$$

where $\mathbf{E}\tau^\alpha \leq 1$.

We get substituting into the multidimensional Garsia-Rodemich-Rumsey inequality

$$\begin{aligned} |\square\xi|(\vec{\delta}) &\leq C_4(\alpha, \vec{\beta}, d) \int_0^{\delta_1} \int_0^{\delta_2} \dots \int_0^{\delta_d} \left[\frac{B}{\prod_{k=1}^d u_k^2} \right]^{1/\alpha} \left[\prod_{k=1}^d \gamma_k u_k^{\gamma_k-1} du_k \right] = \\ &C_5(\alpha, \vec{\beta}, d) K^{1/\alpha} \tau \prod_{k=1}^d \delta_k^{\beta_k/\alpha} \prod_{k=1}^d [\epsilon_k^{-1/\alpha} \delta_k^{-\epsilon_k}]. \end{aligned} \quad (4.13)$$

Choosing $\epsilon_k = C_6(\alpha, \beta_k, k)/|\log \delta_k|$, we arrive to the assertion of theorem 4.2.

Remark 4.2. Let us show the exactness of assertion of theorem 4.2. It is sufficient to consider simple example. Let $d = 1$ and let $\xi(t) = w(t)$, $t \in [0, 1]$ be ordinary Brownian motion. We can choose $\alpha = \alpha(\Delta) := 2 + 2\Delta$, $\beta = \beta(\Delta) := \Delta$, $\Delta = \text{const} \geq 1$. Indeed:

$$\mathbf{E}|w(t) - w(s)|^{2+2\Delta} = C(\Delta) |t - s|^{1+\Delta}, \quad s, t \in [0, 1],$$

Note that

$$\lim_{\Delta \rightarrow \infty} \frac{\beta(\Delta)}{\alpha(\Delta)} = 1/2.$$

But it is well known that

$$\overline{\lim}_{\delta \rightarrow 0+} \frac{\omega(w, \delta)}{\delta^{1/2} |\log \delta|^{1/2}} > 0$$

almost everywhere.

Obtained in this section results specify and generalize ones in the articles [6], [7], [22].

Another approach to the problem of (ordinary) continuity of random fields based on the so-called generic chaining method and entropy technique with described applications see in [2], [4], [11], [13], [19], [28], [29] etc.

5 Concluding remarks.

A. Exactness of our estimations: one dimensional case.

Note that at $\alpha = d$ and following $\alpha = d = 1$ our estimation in the second section and the integer order results from the article [21] coincides up to multiplicative constants. But it is proved in [21] that the integer order estimates are asymptotically exact.

B. Exactness of our estimations: multidimensional case.

Let us prove that the exponents $\vec{\alpha} - 1/p$ in (3.2) - (3.3) are unimprovable. Recall that $p > \max_k(1/\alpha_k)$. As before, $\psi_\alpha(\cdot) = \psi_{(p)}(\cdot)$.

Namely, let us denote in the one-dimensional case $d = 1$

$$V_\alpha(f, \delta) = |\log \omega(f, \delta)| : \left| \log \left[\frac{\delta^\alpha}{\phi(G\psi_{(p)}, \delta)} \cdot \|f\| G\psi_{(p)} \right] \right|, \quad (5.1)$$

$$\underline{V}_\alpha = \inf_{f \in G\psi_{(p)}(\cdot), f \neq \text{const}} \overline{\lim}_{\delta \rightarrow 0+} V_\alpha(f, \delta). \quad (5.2)$$

It follows from theorem 2.1 that $\underline{V}_\alpha \geq 1$; let us prove the opposite inequality. Consider the following example (more precisely, the family of examples):

$$f_{\Delta, \alpha}(x) = x^{\alpha-1/p+\Delta}, \quad \Delta = \text{const} \in (0, 1 - \alpha + 1/p). \quad (5.3)$$

Obviously,

$$\omega(f_{\Delta, \alpha}, \delta) = \delta^{\alpha-1/p+\Delta}.$$

Further, it is no hard to compute using polar coordinates: as $\delta \rightarrow 0+$

$$\frac{\delta^\alpha}{\phi(G\psi_{(p)}, \delta)} \cdot \|f_{\Delta, \alpha}\| G\psi_{(p)} \sim C(\alpha, p, \Delta) \delta^{\alpha-1/p}.$$

Since the value Δ is arbitrary, we conclude $\underline{V}_\alpha \leq 1$.

The multidimensional example may be constructed as a factorable function of a view

$$g_{\Delta, \vec{\alpha}}(\vec{x}) = \prod_{k=1}^d f_{\Delta, \alpha_k}(x_k). \quad (5.4)$$

C. Simplification of our multidimensional estimate.

Let us consider in this paragraph the following important coefficient:

$$L = \prod_{k=1}^d \frac{\alpha_k + 1/p}{\alpha_k - 1/p}, \quad p > 1/\alpha_0, \quad \alpha_0 := \min_k \alpha_k,$$

meaning to extract the main factor as $p \rightarrow 1/\alpha_0$.

We have:

$$L = \prod_{k:\alpha_k > \alpha_0} \frac{\alpha_k + 1/p}{\alpha_k - 1/p} \times \prod_{k:\alpha_k = \alpha_0} \frac{\alpha_k + 1/p}{\alpha_k - 1/p} \leq \prod_{k:\alpha_k > \alpha_0} \frac{\alpha_k + \alpha_0}{\alpha_k - \alpha_0} \times \left\{ \frac{\alpha_0 + 1/p}{\alpha_0 - 1/p} \right\}^M.$$

D. General rectangle distance.

1. Let $X_j = \{x_j\}, j = 1, 2, \dots, d$ be arbitrary sets and $f : Z = \otimes_{j=1}^d X_j \rightarrow R$ be numerical function. Define the following function

$$\rho_f(\vec{x}, \vec{y}) = \rho_f(x, y) = |\square[f](x, y)|, \quad x, y \in Z.$$

Note the following properties of the function $\rho_f(x, y)$.

(a) $\rho_f(x, y) \geq 0$; $\exists j = 1, 2, \dots, d \quad x_j = y_j \Rightarrow \rho_f(x, y) = 0$.

(non-negativity);

(b) $\rho_f(x, y) = \rho_f(y, x)$,

(symmetry);

(c) $\rho_f(x, z) \leq \rho_f(x, y) + \rho_f(y, z)$, $x, y, z \in Z$,

(rectangle inequality).

Definition of a rectangle distance.

Arbitrary numerical function of $2d$ variables $\rho(x, y)$, $x, y \in Z$ which satisfies the properties (a,b,c) is said to be a *rectangle distance*.

Example. Let $\xi = \xi(x)$, $x \in Z$ be a random field with condition

$$\exists q \in [1, \infty], \quad \sup_{x \in Z} [\mathbf{E}|\xi(x)|^q]^{1/q} < \infty.$$

The function

$$\rho^{(\xi, q)}(x, y) = \{\mathbf{E}|\square[\xi](x, y)|^q\}^{1/q}$$

is bounded *natural* rectangle distance generated by r.f. $\xi = \xi(x)$, regarded before.

Obviously, instead the classical L_q norm may be used arbitrary rearrangement norm, for instance, Orlicz, Grand Lebesgue, Lorentz or Marcinkiewicz norm etc.

E. Scaling method.

We intend to prove here the exactness of inequality (1.5) by means of the so-called scaling method, see [26], [27]. Indeed, we can extrapolate the function f in (1.5) as continuous function in the closed interval $[0, 2]$ with support $[0, 2]$ such that on the set $[1, 2]$ f is linear, $f(2) = 0$. For such a function (1.5) also holds.

Introduce the dilation operator $T_\lambda[f] = f(\lambda x)$, $\lambda > 0$. Consider the following strengthening of (1.5) for any continuous function with compact support belonging to the space $C_{\alpha-1/p}^{(0)}$, where by definition the space $C_\beta^{(0)}$, $\beta \in (0, 1]$ consists on all (continuous) function with finite semi-norm

$$\|f\|_{C_\beta^{(0)}} = \sup_{\delta>0} \frac{\omega(f, \delta)}{\delta^\beta} :$$

$$\begin{aligned} \omega(f, \delta) &\leq 8 \cdot 4^{1/p} \cdot \left[\frac{\alpha + 1/p}{\alpha - 1/p} \right] \cdot \delta^{\alpha-1/p} \cdot \gamma(\delta) \cdot \left[\int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p dx dy}{|x - y|^{\alpha p + 1}} \right]^{1/p} =: \\ &\delta^{\alpha-1/p} \cdot \gamma(\delta) \|f\| U(\alpha, p), \end{aligned} \quad (5.5)$$

where $\lim_{\delta \rightarrow 0+} \gamma(\delta) = 0$.

Applying (5.5) for the non-constant function $T_\lambda[f]$, we obtain after simple calculations:

$$\begin{aligned} \|T_\lambda f\|^p U(\alpha, p) &= \int_0^\infty \int_0^\infty \frac{|f(\lambda x) - f(\lambda y)|^p dx dy}{|x - y|^{\alpha p + 1}} = \\ \lambda^{-2} \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p dx dy}{|x/\lambda - y/\lambda|^{\alpha p + 1}} &= \lambda^{-1+\alpha p} \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p dx dy}{|x - y|^{\alpha p + 1}} = \\ \lambda^{-1+\alpha p} \|f\|^p U(\alpha, p), \quad \|T_\lambda f\| U(\alpha, p) &= \lambda^{\alpha-1/p} \|f\| U(\alpha, p); \\ \frac{\omega(f, \lambda \delta)}{(\lambda \delta)^{\alpha-1/p}} &\leq \gamma(\delta) \|f\| U(\alpha, p). \end{aligned} \quad (5.6)$$

We get taking supremum over λ :

$$\|f\|_{C_{\alpha-1/p}^{(0)}} \leq \gamma(\delta) \|f\| U(\alpha, p),$$

which is not true as $\delta \rightarrow 0+$.

F. General Orlicz approach.

Let $\Phi = \Phi(u)$ be again the Young-Orlicz function. We will denote the Orlicz norm by means of the function Φ of a r.v. κ defined on our probabilistic space as $\|\kappa\|_{L(\Phi)}$.

We introduce the natural rectangle distance $\rho_\Phi(x, y)$ as follows:

$$\rho_\Phi(x, y) := \|\square[\xi](x, y), \quad x, y \in D = [0, 1]^d, \quad (5.7)$$

so that for the r.v.

$$Y = \int_D \int_D \Phi \left(\frac{\square[\xi](x, y)}{\rho_\Phi(x, y)} \right) dx dy$$

we have

$$\mathbf{E}Y = \int_D \int_D \mathbf{E}\Phi \left(\frac{\square[\xi](x, y)}{\rho_\Phi(x, y)} \right) dx dy \leq 1, \quad (5.8)$$

since $\int_D \int_D dx dy = 1$.

Let also $\rho^{(\Phi)}(x-y)$ be translation invariant strictly increasing continuous distance majored $\rho_\Phi(x, y)$:

$$\rho_\Phi(x, y) \leq \rho^{(\Phi)}(x - y).$$

We denote the particular distances

$$p_k(|y_k - x_k|) = \rho^{(\Phi)}(1, 1, \dots, 1, |x_k - y_k|, 1, \dots, 1).$$

It follows immediately from the multidimensional version of Garsia-Rodemich-Rumsey inequality that

$$\Omega[\xi](\vec{\delta}) \leq 8^d \int_0^{\delta_1} \int_0^{\delta_2} \dots \int_0^{\delta_d} \Phi^{-1} \left(\frac{4^d Y}{\prod_{k=1}^d u_k^2} \right) \prod_{k=1}^d dp_k(u_k). \quad (5.9)$$

Of course, the inequality (5.9) is pithy if the integral the right-hand side convergent; then the r.f. $\xi(\cdot)$ is continuous with probability one.

The Gaussian (more precisely, subgaussian) case considered in [6], [7], [22] may be obtained by choosing $\Phi(z) = \exp(z^2/2) - 1$. It may be considered easily the example when $\Phi(z) = \exp(|z|^m/m) - 1$, $m = \text{const} > 0$.

G. Fractional Orlicz - Sobolev inequalities.

Let $f : D = [0, 1]^d \rightarrow R$ be (measurable) function. We define the following natural Ψ function depending on the vector positive parameter $\vec{\alpha}$:

$$\tau_{\vec{\alpha}}(p) = \|G_{\vec{\alpha}}[f] \cdot, \cdot\|_{p, D^2, \nu}, \quad p > 1/\min(\alpha_k). \quad (5.10)$$

It will be presumed that the function $\tau_{\vec{\alpha}}(p)$ there exists:

$$\text{supp } \tau_{\vec{\alpha}}(\cdot) = (A, \infty), \quad A > 1/\min \alpha_k.$$

We can construct the following exponential $N = N_{\vec{\alpha}}$ Young-Orlicz function as in remark 1.3:

$$N_{\vec{\alpha}}(u) = e^{[p \log \tau_{\vec{\alpha}}(p)]^* (\log |u|)}, \quad |u| > 3. \quad (5.11)$$

We offer in this subsection a multidimensional (rectangle) version of fractional Orlicz-Sobolev inequality for the exponential Orlicz's space $L(N_{\vec{\alpha}}(\cdot))$. Note that the integer ordinary (interval) Orlicz-Sobolev inequality for the arbitrary Orlicz's space is considered, e.g. in [1], chapter 11; [23], chapter 9.

We infer on the basis of theorem 3.1 and remark 1.3:

Proposition 5.1.

$$\Omega[f](\vec{\delta}) \leq C(\vec{\alpha}, d) \frac{\vec{\delta}^{\vec{\alpha}} \cdot \|f\| L(N_{\vec{\alpha}}(\cdot))}{\phi(G\tau_{\alpha}, \prod_{k=1}^d \delta_k)}. \quad (5.12)$$

As a slight strengthening:

Proposition 5.2.

$$\Omega[f](\vec{\delta}) \leq \inf_{\vec{\alpha}} \left[C(\vec{\alpha}, d) \frac{\vec{\delta}^{\vec{\alpha}} \cdot \|f\| L(N_{\vec{\alpha}}(\cdot))}{\phi(G\tau_{\alpha}, \prod_{k=1}^d \delta_k)} \right]. \quad (5.13)$$

Note in addition that the fundamental function $\phi(L(\Phi), \delta)$ for arbitrary probabilistic Orlicz's spaces $L(\Phi)$ is calculated, e.g. in the classical book of Krasnoselsky M.A., Rutizky Ya.B. [12], chapter 2, section 9:

$$\phi(L(\Phi), \delta) = \delta \cdot \Phi^{-1}(1/\delta). \quad (5.14)$$

See also more modern books [23], [24].

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